# Turbulent Kinetic Equations Characterized by Three and Many Nonequilibrium Degrees of Freedom

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## Abstract

A TURBULENT kinetic equation governing the probability density function of the fluid elements in a turbulent field characterized by three nonequilibrium degrees of freedom was developed in this synoptic. Chung's original kinetic theory of one nonequilibrium degree of freedom was further extended to include three nonequilibrium degrees of freedom. The generalized kinetic equation containing an arbitrary number $N$ of nonequilibrium degrees of freedom also is proposed in this synoptic. General properties of the turbulent kinetic equation developed for multinonequilibrium degrees of freedom were discussed.

## Contents

In 1969, Chung[2] developed his kinetic theory of turbulent chemically reacting flows. He lumped all energy-containing eddies as one nonequilibrium degree of freedom and small fast-fluctuating eddies as many in equilibrium degrees of freedom. He employed the Langevin equation to describe the dynamic change of fluid elements. He assumed that, at high Reynolds number flows, the eddy dynamic process could be approximated as a Markoff process. This assumption was further discussed and justified by Yeh et al.[4] and Durbin.[5] Therefore, there existed a transition probability density function (PDF) between two neighbor phase points in stochastic consideration of fluid-element dynamics in phase space. This transition probability density function was constructed through the solution of Langevin equation by employing Chandrasekhar's Lemma 1.[1] In 1974, Bywater[3] extended Chung's original work to develop a kinetic equation governing the probability density function of the fluid elements for a turbulent field having two significant dynamic-length scales.

### Development of the Formulation of the Kinetic Equation Characterized by Three Nonequilibrium Degrees of Freedom

Following the previous analysis of Chung and Bywater, the instantaneous velocity of a fluid element $e_i(t,x)$ of the present case is decomposed into three components as

$$e_i(t,x) = u_i^1(t,x) + u_i^2(t,x) + u_i^3(t,x) \tag{1}$$

where $u_i^1$, $u_i^2$, and $u_i^3$ are contributions of the three nonequilibrium degrees of freedom to the fluid-element velocity.

Each instantaneous velocity of the fluid element can be divided into the mean and the relative velocities as

$$e_i = e_{0i} + E_i, \qquad u_i^j = u_{0i}^j + U_i^j$$

$$e_{0i} = \sum_j^3 u_{0i}^j, \qquad E_i = \sum_j^3 U_i^j \tag{2}$$

### Transition Probability Density Function

It is supposed that all observable properties of the turbulent field are governed by three nonequilibrium degrees of freedom representing three families of eddies, respectively. The dynamics of nonequilibrium degrees of freedom was described by following three coupled Langevin equations,

$$\frac{\partial U_i^j}{\partial t} = -\beta_{jj}U_i^j - \sum_{k\neq j}^3 \beta_{jk}(U_i^j - U_i^k) + A(t) + K_{ji} \qquad i,j = 1,2,3 \tag{3}$$

The meanings of various terms in Eq. (3) are the same as those for the single Langevin equation employed by Chung except the coupling terms $\beta_{jk}(U_i^j - U_i^k)$, $j \neq k$, $j,k = 1,2,3$; $\beta_{jk}$, $j \neq k$, are the characteristic rate of momentum interaction between the $j$th and the $k$th nonequilibrium degree of freedom.

In order to solve the three coupled Langevin stochastic equations, a mathematical lemma for trivariate Gaussian distribution is developed as Lemma 3 (which was an extension of Chandrasekha's Lemma 2).

From the solution of the coupled Langevin equations, Eq. (3), and employing Lemma 3, the transition probability density function, $\tau$, can be constructed.

### Governing Turbulent Kinetic Equation

Similar to Chung's derivation, the following integral relationship for the joint distribution function between two neighboring phase points can be written as

$$\begin{aligned} &f_3(t + \Delta t, x + \Delta x, U^1, U^1, U^2) \\ &= \int\!\!\int\!\!\int f_3(t,x,U^1 - \Delta U^1, U^2 - \Delta U^2, U^3 - \Delta U^3) \\ &\times \tau(t,x,U^1 - \Delta U^1, U^2 - \Delta U^2, U^3 - \Delta U^3; \\ &\times \Delta U^1, \Delta U^2, \Delta U^3)\, \mathrm{d}\Delta U^1\, \mathrm{d}\Delta U^2\, \mathrm{d}\Delta U^3 \end{aligned} \tag{4}$$

Expanding Eq. (4) into a Taylor series with respect to $\Delta t$, omitting higher-order terms of $\Delta t$, and using the aid of the evaluation of transition moments that appeared in the ex-

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panded series, the kinetic equation of $f_3$ was obtained as

$$\frac{\partial f_3}{\partial t} - \sum_m^3 \sum_j^3 \left( \frac{\partial u_{0m}^j}{\partial t} \frac{\partial f_3}{\partial U_m^j} \right) + \sum_i^3 \left[ \left( e_{0i} + \sum_j^3 U_i^j \right) \frac{\partial f_3}{\partial x_i} \right]$$
$$- \sum_i^3 \left[ \left( e_{0i} + \sum_j^3 U_i^j \right) \left( \sum_k^3 \frac{\partial u_{0m}^k}{\partial x_i} \frac{\partial f_3}{\partial U_m^k} \right) \right]$$
$$= \sum_i^3 \sum_j^3 \frac{\partial}{\partial U_i^j} \left\{ \left[ \beta_{jj} U_i^j + \sum_{j \neq k}^3 \beta_{jk} (U_i^j - U_i^k) - K_{ji} \right] f_3 \right\}$$
$$+ \sum_i^3 \left[ \sum_j^3 \sum_k^3 \frac{\partial^2}{\partial U_i^j \partial U_i^k} (q_3 f_3) \right] \tag{5}$$

Equation (18) is the Fokker-Planck type kinetic equation governing the joint PDF of the fluid element in a turbulence field characterized by three nonequilibrium degrees of freedom. Similar to that was argued in Ref. 3, $q_3$ is defined to satisfy the conservation criterion as

$$q_3 = \left[ \sum_i^3 \sum_j^3 \sum_k^3 \beta_{jk} \langle U_i^j U_i^k \rangle \right] \Big/ 27 \tag{6}$$

### General Properties of the Present Three-Scale Kinetic Equation

The momentum equations obtained from the general moment equation of the present PDF equation, Eq. (18), are identical to the Navier-Stokes equations with the definition of $K_{ji}$ similar to that of Chung's; i.e.,

$$K_{ji} = \sum_m^3 \frac{\nu \partial^2 u_i^j}{\partial x_m \partial x_m} - (1/\rho) \frac{\partial P_0^j}{\partial x_i} \tag{7}$$

where $P_0^j$ are the average pressure associated with the $j$th nonequilibrium degree of freedom and $P_0 = P_0^1 + P_0^2 + P_0^3$. By properly taking moment of Eq. (5), one could obtain

$$\frac{\partial}{\partial t}(e_{0i}) + \sum_k^3 e_{0k} \frac{\partial e_{0i}}{\partial x_k} + \sum_k^3 \frac{\partial}{\partial x_k} \langle E_i E_k \rangle$$
$$= \sum_m^3 \frac{\nu \partial e_{0i}^2}{\partial x_m \partial x_m} - (1/\rho) \frac{\tfrac{1}{3} P_0}{\partial x_i} \tag{8}$$

which is the same as the averaged Navier-Stokes equation.

The present $f_3$ equation also preserved the property of the Gaussian distribution, as was argued by Chung and Bywater, that in the limiting case of the turbulence it would be completely dissipated for times much longer than the characteristic relaxation time of the nonequilibrium eddies. This can be obtained easily by taking the limit of $\beta_{ij} \to \infty$ and solving Eq. (5).

### Generalized $N$ Nonequilibrium Degree-of-Freedom Turbulent Kinetic Equation

If the observable properties of a high Reynolds number turbulent field are governed by arbitrary number $N$ of nonequilibrium degrees of freedom, then the momentum change of the fluid element in the turbulent field is described by the $N$ coupled Langevin equations. The transition PDF of this system can be constructed with the aid of Lemma $N$, which can be obtained by extending the previous Lemma 3. Similar to the previous process, the $f_N$ equation can be obtained as

$$\frac{\partial f_N}{\partial t} - \sum_m^3 \sum_j^N \left( \frac{\partial u_{0m}^j}{\partial t} \frac{\partial f_N}{\partial U_m^j} \right) + \sum_i^3 \left[ \left( e_{0i} + \sum_j^N U_i^j \right) \frac{\partial f_N}{\partial x_i} \right]$$
$$- \sum_i^3 \left[ \left( e_{0i} + \sum_j^N U_i^j \right) \left( \sum_k^N \frac{\partial u_{0m}^k}{\partial x_i} \frac{\partial f_N}{\partial U_m^k} \right) \right]$$
$$= \sum_i^3 \sum_j^N \frac{\partial}{\partial U_i^j} \left\{ \left[ \beta_{jj} U_i^j + \sum_{j \neq k}^N \beta_{jk} (U_i^j - U_i^k) - K_{ji} \right] f_N \right\}$$
$$+ \sum_i^3 \left[ \sum_j^N \sum_k^N \frac{\partial^2}{\partial U_i^j \partial U_i^k} (q_N f_N) \right] \tag{9}$$

where $q_N$ is defined as

$$q_N = \sum_i^3 \sum_j^N \sum_k^N \beta_{jk} \langle U_i^j U_i^k \rangle / (3N^2) \tag{10}$$

Chung's original theory, Bywater's extension, and the previous kinetic equation containing three nonequilibrium degrees of freedom can be obtained by directly substituting $N = 1$, $N = 2$, or $N = 3$ into Eqs. (9) and (10).

## Concluding Remarks

The present analysis derived the kinetic equation of turbulence capable of including more nonequilibrium degrees of freedom. The moment equations derived from the present kinetic equation of multiple nonequilibrium degrees of freedom can be made comparable termwise to those derived from the Navier-Stokes equation.

The generalized kinetic equation of turbulence including an arbitrary number $N$ of nonequilibrium degrees of freedom also was given in the present paper. Other properties of the present multiple nonequilibrium degrees-of-freedom kinetic equation also were discussed, which were consistent with the original arguments of Chung and Bywater.

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